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Some Formulas for the Products of E-Functions and Whittaker Functions

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SOME FORMULAS FOR THE PRODUCTS OF E-FUNCTIONS AND WHITTAKER FUNCTIONS

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Abstract

In this paper a particular integral of a product of Whittaker functions is expressed essentially in terms of another Whittaker function.

Likewise a particular integral of a product of E functions is shown to be essentially an E function.

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1. Introduction

The integral

$$(1)\qquad \int_0^\infty \exp\left[\frac{1}{2}\left(x+\frac{b}{x}\right)\right] W_{k,m}(x)W_{k,m}(b/x)\,dx =$$

$$\pi^{\frac{1}{2}}\,\frac{\Gamma(-k+m)\Gamma(-k-m)}{\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)}\cdot\frac{b^{-\frac{1}{4}}}{2^{2k+1/2}}\,e^{\sqrt{b}}\,W_{2k+\frac{1}{2},2m}(2\sqrt{b}),$$

where $R(k) < -|R(m)|$, $|\text{amp } b| < \frac{3\pi}{2}$, was stated by Erdélyi (see [1], p. 413, (56)).

In this paper we shall establish a generalization of this formula, namely,

$$(2)\qquad \int_0^\infty\cdots\int_0^\infty \exp\left[\frac{1}{2}\left(\frac{b}{x_1,x_2,\ldots,x_{p-1}}\right)\right] W_{k,m}\left(\frac{b}{x_1,x_2,\ldots,x_{p-1}}\right)$$

$$\prod_{r=1}^{p-1}\left\{e^{\frac{1}{2}x_r}W_{k,m}(x_r)\right\}dx_1,\ldots,dx_{p-1} =$$

$$(2\pi)^{\frac{3}{2}p-\frac{3}{2}}\,p^{p+2pk-\frac{3}{2}}\,b^{k}(pb^{1/p})^{\frac{1}{2}-\frac{1}{2}p-pk}$$

$$\times\quad\frac{\Gamma(1-\frac{1}{2}p-pk+pm)\Gamma(1-\frac{1}{2}p-pk-pm)}{\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)^{p}}\,\exp\left[\frac{1}{2}(pb^{1/p})\right]$$

$$\times\quad W_{-\frac{1}{2}+\frac{1}{2}p+pk,pm}(pb^{1/p}),$$

where $R(\frac{1}{2}-k\pm m) > \frac{p-1}{p}$, $b > 0$ and $p = 2,3,4,\ldots$. (2) will be deduced as a particular case from the following formula for MacRobert's E-function which will also be established in this paper, namely,

$$(3)\qquad \int_0^\infty \cdots \int_0^\infty E(\alpha,\beta:: \frac{b}{x_1,x_2,\ldots,x_{p-1}}) \prod_{r=1}^{p-1} \left\{ x_r^{-(r+p)/p} E(\alpha,\beta:: x_r) \right\} dx_1 \cdots dx_{p-1}$$

$$= (2\pi)^{\frac{3}{2}} p^{-\frac{3}{2}} p^{2p-p\alpha-p\beta-\frac{3}{2}} E(p\alpha-p+1, p\beta-p+1:: pb^{1/p}),$$

where $R(\alpha) > (p-1)/p$, $R(\beta) > (p-1)/p$, $b > 0$ and $p = 2,3,4,\ldots$. It may be noted that if we put $p = 2$ in (3), we have

$$(4)\qquad \int_0^\infty x^{-\frac{3}{2}} E(\alpha,\beta::x)E(\alpha,\beta::b/x)dx = \pi^{\frac{3}{2}} 2^{4-2\alpha-2\beta} E(2\alpha-1, 2\beta-1:: 2\sqrt{b}),$$

where $R(\alpha) > \frac{1}{2}$, $R(\beta) > \frac{1}{2}$ and where b is real and positive. The following formula will be required in the proof:

$$(5)\qquad \int_0^\infty \lambda^{-k-1} E(\alpha,\beta::\mu\lambda)d\lambda = \Gamma(k)\Gamma(\alpha-k)\Gamma(\beta-k)\cdot \mu^k ,$$

where $R(\alpha-k) > 0$, $R(\beta-k) > 0$, $R(k) > 0$.

For the definition and properties of the E-functions, the reader is referred to [2], page 352. Formula (3) will be proved in Section 2, while (1) and (2) will be deduced in Section 3.

2. Proof of the multiple integral (3)

Denote the integral on the left of (3) by F(b), then, writing x for x_{p-1}, obtain

$$F(b) = \int_0^\infty \cdots \int_0^\infty \prod_{r=1}^{p-2} \left\{ x_r^{-(r+p)/p} E(\alpha,\beta::x_r) \right\} dx_1,\ldots,dx_{p-2}$$

$$\times \int_0^\infty x^{-(2p-1)/p} E(\alpha,\beta::x) E\left(\alpha,\beta:: \frac{b}{x_1,\ldots,x_{p-2},x}\right) dx.$$

In the last integral, replace x by $\frac{b}{x,x_1,\ldots,x_{p-2}}$ and get

$$b^{1-\frac{1}{p}} F(b) = \int_0^\infty \cdots \int_0^\infty \prod_{r=1}^{p-2} \left\{ x_r^{-(r+1)/p} E(\alpha,\beta::x_r) \right\} dx_1,\ldots,dx_{p-2}$$

$$\times \int_0^\infty x^{-\frac{1}{p}} E(\alpha,\beta::x) E\left(\alpha,\beta:: \frac{b}{x_1,\ldots,x_{p-2},x}\right) dx,$$

so that by differentiation,

$$b^{1-\frac{1}{p}} F'(b) + (1-\tfrac{1}{p}) b^{-\frac{1}{p}} F(b) = \int_0^\infty \cdots \int_0^\infty \prod_{r=1}^{p-2} \left\{ x_r^{-(r+1)/p^{-1}} E(\alpha,\beta::x_r) \right\} dx_1 \ldots dx_{p-2}$$

$$\times \int_0^\infty x^{-\frac{1}{p}-1} F(\alpha,\beta::x) E'\left(\alpha,\beta:: \frac{b}{x_1,\ldots,x_{p-2}x}\right) dx,$$

and

$$b^{1-\frac{1}{p}} F''(b) + 2(1-\tfrac{1}{p}) b^{-\frac{1}{p}} F'(b) + (1-\tfrac{1}{p})(-\tfrac{1}{p}) b^{-\frac{1}{p}-1} F(b)$$

$$= \int_0^\infty \cdots \int_0^\infty \prod_{r=1}^{p-2} \left\{ x_r^{-(r+1)/p^{-2}} E(\alpha,\beta::x_r) \right\} dx_1 \ldots dx_{p-2}$$

$$\times \int_0^\infty x^{-\frac{1}{p}-2} E(\alpha,\beta::x) E''\left(\alpha,\beta:: \frac{b}{x_1,\ldots,x_{p-2}x}\right) dx.$$

Now, remembering that the differential equation

$$z^2 w'' - z(z+\alpha+\beta-1)w' + \alpha\beta w = 0 \tag{6}$$

has solutions

$$F(\alpha,\beta::z), \quad z^\alpha\, {}_1F_1(\alpha;1+\alpha-\beta;z) \quad \text{and} \quad z^\beta\, {}_1F_1(\beta;1+\beta-\alpha;z),$$

we can write the last result as

$$b^{3-\frac{1}{p}} F''(b) + 2(1-\tfrac{1}{p})b^{2-\frac{1}{p}} F'(b)+(1-\tfrac{1}{p})(-\tfrac{1}{p})b^{1-\frac{1}{p}} F(b)$$

$$= \int_0^\infty \cdots \int_0^\infty \prod_{r=1}^{p-2} \left\{ x_r^{-(r+1)/p} E(\alpha,\beta::x_r) \right\} dx_1,\ldots,dx_{p-2}$$

$$\times \int_0^\infty x^{-\frac{1}{p}} E(\alpha,\beta::x) \left\{ \frac{b}{x_1,\ldots,x_{p-2},x} \left(\frac{b}{x_1,\ldots,x_{p-2},x} + \alpha+\beta-1 \right) \right.$$

$$\left. E' - \alpha\beta E \right\} dx,$$

where

$$E \equiv E\left(\alpha,\beta:: \frac{b}{x_1,\ldots,x_{p-2},x}\right).$$

Therefore

$$b^{3-\frac{1}{p}} F'''(b) + 2(1-\tfrac{1}{p})b^{2-\frac{1}{p}} F'(b) + (1-\tfrac{1}{p})(-\tfrac{1}{p})b^{1-\frac{1}{p}} F(b) - (\alpha+\beta-1)b^{2-\frac{1}{p}}$$

$$F'(b) - (\alpha+\beta-1)b^{1-\frac{1}{p}} F(b) + \alpha\beta b^{1-\frac{1}{p}} F(b) = L,$$

where

$$L = b^2 \int_0^\infty \cdots \int_0^\infty \prod_{r=1}^{p-2} \left\{ x_r^{-(r+1)/p-2} E(\alpha,\beta::x_r) \right\} dx_1,\ldots,dx_{p-2}$$

$$\times \int_0^\infty x^{-\frac{1}{p}-2} E(\alpha,\beta::x) E'(\alpha,\beta:: \frac{b}{x_1,\ldots,x_{p-2},x}) \, dx.$$

To find L, we go back to the original definition of F(b), namely,

$$F(b) = \int_0^\infty \cdots \int_0^\infty \prod_{r=1}^{p-1} \left\{ x_r^{-\frac{r}{p}-1} E(\alpha,\beta::x_r) \right\} dx_1,\ldots,dx_{p-1} \; E(\alpha,\beta:: \frac{b}{x_1,\ldots,x_{p-1}})$$

so that

$$F'(b) = \int_0^\infty \cdots \int_0^\infty \prod_{r=1}^{p-1} \left\{ x_r^{-\frac{r}{p}-2} E(\alpha,\beta::x_r) \right\} dx_1,\ldots,dx_{p-1}$$

$$\times E'(\alpha,\beta:: \frac{b}{x_1,\ldots,x_{p-1}}).$$

If we write $xx_1x_2\cdots x_{p-2}$ instead of $x_1x_2\cdots x_{p-1}$, we have

$$F'(b) = \int_0^\infty \cdots \int_0^\infty \prod_{r=1}^{p-2} \left\{ x_r^{-(r+1)/p-2} E(\alpha,\beta::x_r) \right\} dx_1,\ldots,dx_{p-2}$$

$$\times \int_0^\infty x^{-\frac{1}{p}-2} E(\alpha,\beta::x) E'(\alpha,\beta:: \frac{b}{x_1,\ldots,x_{p-2}x}) \, dx,$$

so that

$$L = b^2 F'(b).$$

If we substitute for L, we can write the last result as

$$(7) \quad b^{3-\frac{1}{p}} F''(b) + \left[(1-\tfrac{1}{p}) b^{2-\frac{1}{p}} - b^2 - (\alpha+\beta+\tfrac{1}{p}-2) b^{2-\frac{1}{p}} \right] F'(b)$$
$$+ b^{1-\frac{1}{p}} \left[\alpha\beta - (\alpha+\beta)(1-\tfrac{1}{p}) + (1-\tfrac{1}{p})^2 \right] F(b) = 0.$$

If in (6), we put $z = pb^{1/p}$, and replace α, β by $p\alpha-p+1$, $p\beta-p+1$, we obtain (7) with w in place of F(b), and z in place of b. It follows that for some A and B independent of b we must have

$$F(b) = A(pb^{1/p})^{p\alpha-p+1} \, {}_1F_1(p\alpha-p+1;1+p\alpha-p\beta;pb^{1/p})$$
$$+ B(pb^{1/p})^{p\beta-p+1} \, {}_1F_1(p\beta-p+1;1+p\beta-p\alpha;pb^{1/p}).$$

Now F(b) is symmetrical in α and β, and so are the coefficients of A and B. Therefore, if $A = f(\alpha,\beta)$, it follows that $B = f(\beta,\alpha)$. Again, if we substitute for F(b), we have

$$b^{1-\frac{1}{p}}F(b) = \int_0^\infty \cdots \int_0^\infty \prod_{r=1}^{p-2} \left\{ x_r^{-(r+1)/p} E(\alpha,\beta::x_r) \right\} dx_1,\ldots,dx_{p-2}$$

$$\times \int_0^\infty x^{-\frac{1}{p}} F(\alpha,\beta::x) E(\alpha,\beta:: \frac{b}{x_1,\ldots,x_{p-2}x}) dx$$

$$= A(p)^{p\alpha-p+1} b^{\alpha}\ {}_1F_1(p\alpha-p+1, 1+p\alpha-p\beta; pb^{1/p})$$

$$+ B(p)^{p\beta-p+1} b^{\beta}\ {}_1F_1(p\beta-p+1, 1+p\beta-p\alpha; pb^{1/p}).$$

Here, expand the last E-function in the last expression by means of the formula ([2], p.353)

$$(8) \qquad E(\alpha,\beta::z) = \sum_{\alpha,\beta} \Gamma(\beta-\alpha)\Gamma(\alpha)\, z^{\alpha}\ {}_1F_1(\alpha; 1+\alpha-\beta; z)$$

and let it be assumed that $\alpha < \beta$; multiply by $b^{-\alpha}$ and let $b \to 0$; then as $\frac{1}{p} + \beta - \alpha > 0$,

$$Ap^{p\alpha-p+1} = \int_0^\infty \cdots \int_0^\infty \prod_{r=1}^{p-2} \left\{ x_r^{-(r+1)/p-\alpha} E(\alpha,\beta::x_r) \right\} dx_1,\ldots,dx_{p-2}$$

$$\times \int_0^\infty x^{-\frac{1}{p}-\alpha} E(\alpha,\beta::x)dx\ \Gamma(\alpha)\Gamma(\beta-\alpha).$$

The repeated integral here is a product of simple integrals, all of which are given by (5). Thus

$$Ap^{p\alpha-p+1} = \prod_{r=1}^{p-2} \left\{ \Gamma(\tfrac{1}{p}+\alpha-1+\tfrac{r}{p})\Gamma(1-\tfrac{1}{p}-\tfrac{r}{p})\Gamma(1+\beta-\alpha-\tfrac{1}{p}-\tfrac{r}{p}) \right\}$$

$$\times \Gamma(\tfrac{1}{p}+\alpha-1)\Gamma(1-\tfrac{1}{p})\Gamma(1+\beta-\alpha-\tfrac{1}{p})\Gamma(\alpha)\Gamma(\beta-\alpha)$$

$$= (2\pi)^{\frac{3}{2}p-\frac{3}{2}}\, p^{-\frac{1}{2}+p-p\beta}\, \Gamma(p\alpha-p+1)\Gamma(p\beta-p\alpha),$$

where we have used the well-known equation

$$\Gamma(z)\Gamma(z+\tfrac{1}{p}) \cdots \Gamma(z+\tfrac{p-1}{p}) = (2\pi)^{\frac{1}{2}p-\frac{1}{2}} p^{\frac{1}{2}-pz} \Gamma(pz),$$

to simplify the right hand side. Thus

$$A = (2\pi)^{\frac{3}{2}p-\frac{3}{2}} p^{2p-p\alpha-p\beta-\frac{3}{2}} \Gamma(p\alpha-p+1)\Gamma(p\beta-p\alpha),$$

and consequently,

$$B = (2\pi)^{\frac{3}{2}p-\frac{3}{2}} p^{2p-p\beta-p\alpha-\frac{3}{2}} \Gamma(p\beta-p+1)\Gamma(p\alpha-p\beta).$$

Thus from (8), the proof of (3) is complete.

3. Particular cases

In (3) and (4), put $\alpha = \frac{1}{2} - k + m$, $\beta = \frac{1}{2} - k - m$ and use the equation ([2] p. 351):

$$(9)\quad E(\tfrac{1}{2} - k+m, \tfrac{1}{2} -k-m::z) = \Gamma(\tfrac{1}{2} -k+m)\Gamma(\tfrac{1}{2} -k-m)z^{-k}e^{\frac{1}{2}z} W_{k,m}(z)$$

and so obtain (2) and (1).

REFERENCES

[1] Erdélyi, A. - Tables of integral transforms, II; New York, McGraw-Hill, 1954.

[2] MacRobert, T.M. - Functions of a complex variable; New York, MacMillan, 4th Edit., 1954.

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